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A tensorial theory of the higher-order constants of motion in quantum mechanics

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Abstract. A general theory of quantum mechanical constants of motion of arbitrary order in the momenta is developed. Because of rotational invariance the theory assumes an elegant tensorial form. The fundamental equations for the nth-order constants of motion in m variables are set up in Cartesian coordinates. The first two equations of the set do not contain the potential function and their solutions are shown to be polynomials of degree n and $n-1$ with coefficients satisfying certain symmetry relations. The next two equations contain the first and the second derivatives of the potential function. From these equations all unknown functions are eliminated in the special cases $n=3$ and $n=4$. The elimination gives partial differential equations for the potential function, of second and third order for $n=3$, and third and fourth order for $n=4$. The possibility of extending the results to higher values of n is indicated.

To illustrate the use of the theory the 'anisotropic oscillator' in two dimensions is discussed in detail, and all its constants of motion of second and third order are determined. They correspond to the frequency ratios 2 : 1 and 3 : 1 only. There is only one constant of motion, of third order, for the frequency ratio 3 : 1. For the frequency ratio 2 : 1 there are three constants of motion and they do not appear to generate a Lie algebra.

1. Introduction

It is well known (Lenz 1924, Pauli 1926, Fock 1935, Bargmann 1936, Park 1960, Bethe and Leon 1962, Swamy and Biedenharn 1964, Dothan *et al* 1965, Barut *et al* 1966, Flamand 1966, Bander and Itzykson 1966, Hughes 1967, Joseph 1967, Barut and Kleinert 1967, Majumdar and Basu 1974) that the so-called 'accidental degeneracy' of the energy levels of the hydrogen atom with regard to the orbital quantum number is caused by an operator (Englefield 1972 § 3.3)

$$\boldsymbol{M}=(2\mu)^{-1}(\boldsymbol{p}\times\boldsymbol{L}-\boldsymbol{L}\times\boldsymbol{p})-Z\,\mathrm{e}^2r^{-1}\boldsymbol{r} \tag{1}$$

commuting with the Hamiltonian $H=(2\mu)^{-1}p^2-Z\,\mathrm{e}^2/r$. $\boldsymbol{M}$ is an additional constant of motion of second order in the momenta for the one-centre Coulomb problem. The components of $\boldsymbol{L}$ (the angular momentum) and $\boldsymbol{M}$ obey the commutation rules for the generators of O(4) if the energy is negative and of O(3, 1) if the energy is positive, and act as raising and lowering operators on the states of the atom. In the hope of obtaining results of this kind having group-theoretical implications, many authors (Eisenhart 1934, 1948, Fris *et al* 1965, Winternitz *et al* 1966, Makarov *et al* 1967,

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Havas 1975a, b, Dietz 1976) have studied the one-particle Schrödinger equation in two and three dimensions, and have determined all second-order constants of motion and the corresponding potentials. These investigations have revealed an intimate connection between the existence of quadratic constants of motion and the separability of the Schrödinger equation. It has been found that a quadratic constant of motion can exist in the two-dimensional case if and only if the Schrödinger equation separates in one of the four coordinate systems in which the Helmholtz equation separates. In three dimensions separability is found to be equivalent to the existence of two quadratic constants of motion. Generally, one can expect that, in m dimensions, the separability would imply the existence of $m-1$ quadratic constants of motion. Strictly speaking, the result is not new and occurs in the investigations of Stäckel (1890, 1891, 1893), Levi-Civita (1904), Dall'Aqua (1908, 1912) and Burgatti (1911) on the Hamilton–Jacobi equation. Stäckel was the first to point out the connection between separability and constants of motion.

While the second-order constants of motion in two and three dimensions have been studied in detail, little work has been done on constants of motion of higher orders and in higher dimensions. The only exceptions are the isotropic harmonic oscillator, the anisotropic harmonic oscillator with rational frequency ratios, the one- and two-centre Coulomb problems, and the work of Majumdar and Englefield (1977, to be referred to as I) on third-order constants of motion. In the m-dimensional Coulomb problems the only constant of motion so far determined is the analogue of the operator $\boldsymbol{M}$. The object of the present paper is partially to fill this gap by developing a general theory of nth-order constants of motion in m variables.

2. The fundamental equations

We consider an operator of the type

$$L=\sum_{r=0}^{n}\phi^{r}_{i_1\ldots i_r}\cdot p_{i_1}\ldots p_{i_r} \tag{2}$$

commuting with the m-dimensional Hamiltonian

$$H=-\tfrac{1}{2}\nabla^2_m+V(x_1,\ldots,x_m), \tag{3}$$

where $p_i=\partial/\partial x_i$ and ϕ^r are unknown functions of m Cartesian coordinates $x_1,\ldots,x_m$. Evaluating the commutator $[H, L]$ and separately equating to zero the coefficients of all distinct products of the p's we have the following system of partial differential equations for ϕ^r:

$$\tfrac{1}{2}\nabla^2\phi^k+\frac{1}{k}\sum_{r=1}^{k}\partial_{i_r}\phi_{i_1\ldots i_{r-1}i_{r+1}\ldots i_k}+\sum_{s=k+1}^{n}{}^{s}\mathrm{C}_k\phi^{s}\partial_{i_{k+1}\ldots i_s}V=0 \tag{4}$$

where $k=0,\ldots,n+1$, $\phi^k=\phi_{i_1\ldots i_k}$, $\phi^s=\phi_{i_1\ldots i_s}$, $\partial_k=\partial/\partial x_k$, $\partial_{kl}=\partial^2/\partial x_k\partial x_l,\ldots,$ the third term is non-existent for $k=n$, $n+1$ and the first term for $k=n+1$, and the dummy suffix notation is used throughout.

Equation (4) is the fundamental equation of the problem. It is already in tensorial form and becomes a true tensor equation when the ordinary derivatives are replaced by covariant derivatives. Written more explicitly, the equations for the various values of k

are

$$\sum_r \partial_{i_r}\phi_{i_1\ldots i_{r-1}i_{r+1}\ldots i_{n+1}} = 0 \tag{5.1}$$

$$\tfrac{1}{2}\nabla^2\phi^n + \frac{1}{n}\sum_r \partial_{i_r}\phi_{i_1\ldots i_{r-1}i_{r+1}\ldots i_n} = 0 \tag{5.2}$$

$$\tfrac{1}{2}\nabla^2\phi^{n-1} + \frac{1}{n-1}\sum_r \partial_{i_r}\phi_{i_1\ldots i_{r-1}i_{r+1}\ldots i_{n-1}} + {}^nC_1\phi^n\partial_{i_n}V = 0 \tag{5.3}$$

$$\tfrac{1}{2}\nabla^2\phi^{n-2} + \frac{1}{n-2}\sum_r \partial_{i_r}\phi_{i_1\ldots i_{r-1}i_{r+1}\ldots i_{n-2}} + {}^{n-1}C_1\phi^{n-1}\partial_{i_{n-1}}V + {}^nC_2\phi^n\partial_{i_{n-1}i_n}V = 0 \tag{5.4}$$

$$\vdots$$

$$LV = H\phi^0. \tag{5.n+2}$$

Equation (5.1) represents ${}^{m+n}C_{n+1}$ first-order partial differential equations satisfied by ${}^{m+n-1}C_n$ functions ϕ^n. It will be shown in § 3 that ϕ^n, ϕ^{n-1} are polynomials in $x_1, \ldots, x_m$ of degree n and $n-1$ respectively. After these functions have been determined by solving equations (5.1) and (5.2), one is faced with the much harder task of solving the remaining equations, which involve the potential function V. Some reductions of these equations to more manageable forms are obtained in § 4 but not much progress is made towards their solution.

3. The nature of solutions of the equations independent of V

To obtain some idea of the solutions of equation (5.1) we consider the set of equations

$$\begin{aligned}
&\partial_1\phi_{1\ldots1} = 0,\\
&\partial_i\phi_{1\ldots1} + n\partial_1\phi_{1\ldots1i} = 0,\\
&\partial_j\phi_{1\ldots1i} + \partial_i\phi_{1\ldots1j} + (n-1)\partial_1\phi_{1\ldots1ij} = 0,\\
&\partial_k\phi_{1\ldots1ij} + \partial_i\phi_{1\ldots1jk} + \partial_j\phi_{1\ldots1ki} + (n-2)\partial_1\phi_{1\ldots1ijk} = 0,\\
&\vdots\\
&\qquad (i, j, k, \ldots \neq 1).
\end{aligned}$$

Integrating these equations from the top, we find that $\phi_{1\ldots1} = f$, $\phi_{1\ldots1i} = x_1 g + h$, $\phi_{1\ldots1ij} = x_1^2 p + x_1 q + r, \ldots$, where $f, g, h, p, q, r, \ldots$ are functions of $x_2, \ldots, x_m$. Thus ϕ^n is a polynomial in x_1 of degree not exceeding the number of missing 1's amongst the indices of ϕ^n, the coefficients of the polynomial involving the other variables $x_2, \ldots, x_m$. Since all indices are equivalent in (5.1), it follows that ϕ^n is a polynomial in all the variables with the above restriction on their powers. It will be seen later that the degree of the polynomial cannot be higher than n, but the result just stated enables us to make a start and write a particular solution of (5.1) in the form

$$\phi_{i_1\ldots i_n} = (i_1 \ldots i_n \mid a_1 \ldots a_q)x_{a_1} \ldots x_{a_q}.$$

The coefficients on the right-hand side are the components of a tensor of rank $n+q$ symmetric to all permutations of the subscripts of type i and also to all permutations of those of type a. Substituted in (5.1), this leads to a system of linear homogeneous equations for the coefficients $(i \mid a)$. These algebraic equations are equivalent to the system (5.1).

The coefficient tensor introduced above is intimately related to the derivative tensor

$$\partial_{a_1 \ldots a_q} \phi_{i_1 \ldots i_n},$$

and becomes a multiple of the latter when the order of the derivatives equals the degree of the polynomial solution of (5.1). We shall now investigate the symmetries of these tensors, first for $n = 3$ and then for arbitrary n. For $n = 3$, equations (5.1) have the form

$$\partial_l \phi_{ijk} + \partial_i \phi_{jkl} + \partial_j \phi_{kli} + \partial_k \phi_{lij} = 0.$$

Differentiating this 'cyclic relation' once or twice and choosing i, j, k, l and the variables of differentiation suitably, we have

$$\begin{aligned} &\partial_b (\partial_a \phi_{ijk} + \partial_i \phi_{jka} + \partial_j \phi_{kia} + \partial_k \phi_{ija}) = 0 \\ &\partial_a (\partial_b \phi_{ijk} + \partial_i \phi_{jkb} + \partial_j \phi_{kib} + \partial_k \phi_{ijb}) = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} &\partial_i (\partial_b \phi_{jka} + \partial_a \phi_{jkb} + \partial_k \phi_{jab} + \partial_j \phi_{kab}) = 0 \\ &\partial_j (\partial_b \phi_{kia} + \partial_a \phi_{kib} + \partial_k \phi_{iab} + \partial_i \phi_{kab}) = 0 \end{aligned} \tag{7}$$

$$\partial_{jk} (\partial_a \phi_{ibc} + \partial_b \phi_{ica} + \partial_c \phi_{iab} + \partial_i \phi_{abc}) = 0. \tag{8}$$

Subtracting twice equation (6) from the six equations obtained by the cyclic permutation of i, j, k in (7), we have

$$\partial_{ab} \phi_{ijk} = \partial_{ij} \phi_{kab} + \partial_{jk} \phi_{iab} + \partial_{ki} \phi_{jab}. \tag{9.1}$$

Differentiating this with respect to x_c gives

$$\partial_{abc} \phi_{ijk} = \partial_{ijc} \phi_{kab} + \partial_{jkc} \phi_{iab} + \partial_{kic} \phi_{jab}. \tag{9.2}$$

(9.1) and (9.2) are two of the symmetry relations of the derivative tensor for $n = 3$. Another relation is obtained by adding together all the equations arising from the cyclic permutation of i, j, k and a, b, c in equations (8) and (9.2). The result is

$$\partial_{abc} \phi_{ijk} = -\partial_{ijk} \phi_{abc}. \tag{10}$$

It can be proved by induction that the general result for arbitrary n is

$$\partial_{a_1 \ldots a_q} \phi_{i_1 \ldots i_n} = \sum \partial_{i_{p_1} \ldots i_{p_r} a_{r+1} \ldots a_q} (-1)^r \phi_{i_{p_{r+1}} \ldots i_{p_n} a_1 \ldots a_r}, \tag{11}$$

where the sum is over all r-combinations of the i's with fixed $a_1, \ldots, a_r$. For $r = q = n$ the summation on the right-hand side of (11) disappears and the symmetry relation takes the elegant form

$$\partial_{a_1 \ldots a_n} \phi_{i_1 \ldots i_n} = (-1)^n \partial_{i_1 \ldots i_n} \phi_{a_1 \ldots a_n}. \tag{12}$$

We shall now prove an important theorem concerning the nature of solutions of (5.1). The above derivation of the symmetries, clearly, remains valid for a tensor with $q > n$. Taking $q = n + 1$, we have

$$\begin{aligned} &\partial_{a_1} (\partial_{a_2 \ldots a_{n+1}} \phi_{i_1 \ldots i_n}) \\ &\quad = (-1)^n \partial_{a_1} (\partial_{i_1 \ldots i_n} \phi_{a_2 \ldots a_{n+1}}) = (-1)^n \partial_{i_1} (\partial_{a_1 i_2 \ldots i_n} \phi_{a_2 \ldots a_{n+1}}) \\ &\quad = \partial_{i_1 a_2 \ldots a_{n+1}} \phi_{a_1 i_2 \ldots i_n}. \end{aligned} \tag{13}$$

Thus, the tensor is unaffected by the interchange of an index of ϕ^n with an index of differentiation. Combining this new symmetry with the symmetry already existing in the i's and the a's, we see that the tensor remains unaffected by any permutation of its $2n+1$ indices. The components of this totally symmetric tensor, the $(n+1)$th derivatives of the ϕ^n, must all vanish by virtue of the cyclic relations. This proves the result stated earlier that the ϕ^n are polynomials in $x_1, \ldots, x_m$ of degree not higher than n. A similar result holds for the inhomogeneous equations (5.2). After $n-1$ differentiations the inhomogeneous terms involving the ϕ^n vanish and the equations take the same form as in (13) with n replaced by $n-1$. The ϕ^{n-1} are, therefore, polynomials in the x's of degree not higher than $n-1$.

Since the coefficient tensor is a multiple of the derivative tensor the two tensors must possess the same symmetries. The symmetries of the coefficient tensor corresponding to (11) and (12) are

$$(i_1 \ldots i_n \mid a_1 \ldots a_q) = (-1)^r \sum_{i_{p_1}, \ldots, i_{p_r}} (i_{p_{r+1}} \ldots i_{p_n} a_1 \ldots a_r \mid i_{p_1} \ldots i_{p_r} a_{r+1} \ldots a_q) \tag{14}$$

and

$$(i_1 \ldots i_n \mid a_1 \ldots a_n) = (-1)^n (a_1 \ldots a_n \mid i_1 \ldots i_n). \tag{15}$$

In (14) the summation is over all r-combinations of the i's with fixed $a_1, \ldots, a_r$. The relation (15) means that the coefficient of $x_{a_1} \ldots x_{a_n}$ in the expression for $\phi_{i_1 \ldots i_n}$ is equal in magnitude to the coefficient of $x_{i_1} \ldots x_{i_n}$ in the expression for $\phi_{a_1 \ldots a_n}$. The two coefficients are of the same sign if n is even and of opposite signs if n is odd. Relation (14) enables one to express every coefficient in a ϕ^n as a sum of coefficients of terms of the same degree in other ϕ^n's. The coefficients in the expressions for ϕ^n available in the literature have been found to obey these symmetries in every individual case tested. The case $n=3$, $m=2$, worked out in I, is discussed from this viewpoint in appendix 2.

We conclude this section by giving an alternative proof of the theorem on the nature of solutions of (5.1) and (5.2). Since these equations do not depend on V, some simplification can be effected by setting $V=0$, $H=-\frac{1}{2}\nabla^2$. Now, the most general operator commuting with the m-dimensional Laplacian is an arbitrary polynomial in the momenta p_k and the angular momenta $M_{ij} = x_i p_j - x_j p_i$ ($i<j$; $i, j = 1, \ldots, m$). Using the abbreviation M_l ($l = 1, \ldots, \frac{1}{2}m(m-1)$) for the M_{ij} taken in any order, the operator can be written as

$$L^0 = \sum_n L_n, \qquad L_n = \sum_k A^{n-k}_{i_1 \ldots i_n} M_{i_1} \ldots M_{i_{n-k}} p_{i_{n-k+1}} \ldots p_{i_n}, \tag{16}$$

where the coefficients $A^{n-k}_{i_1 \ldots i_n}$ are symmetric to all permutations of the last k subscripts but are otherwise arbitrary. The highest power of the x's in this expansion is n and occurs in the term with $k=0$. As k increases the power of the x's decreases in steps of unity until it becomes zero in the last term with $k=n$. Thus, the ϕ^n's are again seen to be polynomials of degree n in the x's. Similar considerations apply to equations (5.2).

4. Differential equations satisfied by V

From a study of the cases $n=2$, $m=3$ and $n=3$, $m=2$, it appears that the elimination of ϕ^{n-2} and ϕ^{n-3} from (5.3) and (5.4), in the general case, will lead to partial differential equations of nth and $(n-1)$th order for V. The conjecture is found to be correct for $n=3, 4$ and arbitrary m. The equations for these cases will be derived here in much the same way as the symmetries of § 3 were derived. The same linear combinations of the differentiated cyclic relations are taken in both cases, but, instead of zeros, there occur, in the present case, expressions involving V on the right-hand side of the relations. The method works for any n, but the complexity of the calculations increases enormously as n increases.

With

$$2T_{ij} = -\nabla^2\phi_{ij} - 6\phi_{ij\alpha}\partial_\alpha V, \tag{17}$$

equations (5.3) and (5.4), for $n = 3$, can be written as

$$\partial_j\phi_i + \partial_i\phi_j = 2T_{ij}, \tag{18}$$

$$\partial_i\phi = -\tfrac{1}{2}\nabla^2\phi_i - 2\phi_{i\alpha}\partial_\alpha V - 3\phi_{i\alpha\beta}\partial_{\alpha\beta}V. \tag{19}$$

By the theorem of § 3, the ϕ_{ijk} are cubic and ϕ_{ij} are quadratic expressions in the x's containing some arbitrary coefficients. Differentiating (18) with respect to x_k and permuting i, j, k cyclically, we have

$$\partial_{kj}\phi_i + \partial_{ki}\phi_j = 2\partial_k T_{ij}, \tag{20.1}$$

$$\partial_{ji}\phi_k + \partial_{jk}\phi_i = 2\partial_j T_{ki}, \tag{20.2}$$

$$\partial_{ik}\phi_j + \partial_{ij}\phi_k = 2\partial_i T_{jk}, \tag{20.3}$$

whence

$$\partial_{jk}\phi_i = \partial_k T_{ij} + \partial_j T_{ki} - \partial_i T_{jk}. \tag{21}$$

Partial differential equations for V are obtained by equating to zero the expression for $\partial_l(\partial_{jk}\phi_i) - \partial_j(\partial_{kl}\phi_i)$. The equations are third order and of the form

$$S_{ik:jl} \equiv \partial_{kl}T_{ij} - \partial_{il}T_{jk} - \partial_{jk}T_{il} + \partial_{ij}T_{lk} = 0. \tag{22}$$

The tensor $S_{ik:jl}$ is antisymmetric to the interchange of subscripts within a partition but symmetric to the interchange of partitions. It is therefore of the same symmetry as the Riemann tensor. It also satisfies the Bianchi identities

$$\partial_m S_{ik:jl} + \partial_i S_{km:jl} + \partial_k S_{mi:jl} = 0, \tag{23}$$

and the cyclic relations

$$S_{ik:jl} + S_{kj:il} + S_{ji:kl} = 0. \tag{24}$$

In m-dimensional space, the Riemann tensor has $\frac{1}{12}m^2(m^2-1)$ independent components and there are as many differential equations of the type (22) for V. Since the first term of the expression (17) for T_{ij} vanishes on differentiation, equations (22), with $T_{ij} = \partial_{ij\alpha}\partial_\alpha V$, assume the form

$$\partial_{kl}(\phi_{ij\alpha}\partial_\alpha V) - \partial_{il}(\phi_{jk\alpha}\partial_\alpha V) - \partial_{kj}(\phi_{il\alpha}\partial_\alpha V) + \partial_{ij}(\phi_{lk\alpha}\partial_\alpha V) = 0. \tag{25}$$

A second set of partial differential equations for V is obtained by eliminating ϕ from equation (19) and using expression (21) (with $j = k = \alpha$) for $\nabla^2\phi_i$. Due to cancellation of all third-order derivatives of V the process results in second-order equations of the form

$0 = 2\partial_j(\phi_{i\alpha}\partial_\alpha V) - 3\partial_j(\partial_\beta\phi_{i\alpha\beta} \,.\, \partial_\alpha V)$ minus the same expression with i, j interchanged. (26)

Cancellation of derivatives of the highest order seems to be a general feature of equations of this type and is likely to occur for any value of n.

The various points mentioned above are better illustrated by taking $n=4$. In this case (5.3) and (5.4) become

$$\partial_k\phi_{ij}+\partial_i\phi_{jk}+\partial_j\phi_{ki}=2T_{ijk}, \qquad \text{with } 2T_{ijk}=-\tfrac{3}{2}\nabla^2\phi_{ijk}-12\phi_{ijk\alpha}\partial_\alpha V \tag{27}$$

$$\partial_j\phi_i+\partial_i\phi_j=-\nabla^2\phi_{ij}+U_{ij}, \qquad \text{with } U_{ij}=-6\phi_{ij\alpha}\partial_\alpha V-12\phi_{ij\alpha\beta}\partial_{\alpha\beta}V. \tag{28}$$

Here ϕ_{ijkl} are quartic and ϕ_{ijk} cubic functions of the x's. Differentiation of (27) yields the relations

$$\partial_b(\partial_a\phi_{ij}+\partial_i\phi_{ja}+\partial_j\phi_{ia})=2\partial_bT_{ija}, \tag{29.1}$$

$$\partial_a(\partial_b\phi_{ij}+\partial_i\phi_{jb}+\partial_j\phi_{ib})=2\partial_aT_{ijb}, \tag{29.2}$$

$$\partial_j(\partial_i\phi_{ab}+\partial_b\phi_{ia}+\partial_a\phi_{ib})=2\partial_jT_{abi}, \tag{29.3}$$

$$\partial_i(\partial_j\phi_{ab}+\partial_b\phi_{ja}+\partial_a\phi_{jb})=2\partial_iT_{abj}. \tag{29.4}$$

Subtracting the last two equations of this set from the first two, we obtain

$$\partial_{ab}\phi_{ij}-\partial_{ij}\phi_{ab}=\partial_bT_{ija}+\partial_aT_{ijb}-\partial_jT_{abi}-\partial_iT_{abj}. \tag{30}$$

In the case of the homogeneous equations, $\partial_k\phi_{ij}+\partial_i\phi_{jk}+\partial_j\phi_{ki}=0$, for $n=2$, zeros occur instead of T's on the right-hand side of equations (27), (29) and (30), and relation (30) reduces to $\partial_{ab}\phi_{ij}=\partial_{ij}\phi_{ab}$. Expressions for the third-order derivatives of ϕ_{ij} are now obtained by differentiating and combining equations of the type (27) and (30). Thus

$$\begin{aligned}3\partial_{abc}\phi_{ij}&=2\partial_a(\partial_{bc}\phi_{ij}-\partial_{ij}\phi_{bc})+\partial_i(\partial_{aj}\phi_{bc}-\partial_{bc}\phi_{aj})\\&\quad+\partial_j(\partial_{ai}\phi_{bc}-\partial_{bc}\phi_{ai})+\partial_{bc}(\partial_a\phi_{ij}+\partial_i\phi_{ja}+\partial_j\phi_{ai})\\&=2\partial_{bc}T_{ija}+\tfrac{2}{3}\partial_{ij}T_{abc}-\partial_{ja}T_{bci}-\partial_{ia}T_{bcj}\end{aligned}$$

plus similar terms arising from cyclic permutation of a, b, c. (31)

This is symmetric to the interchange of i with j and to all permutations of a, b, c. Fourth-order equations for V now follow from the vanishing of the expression for $\partial_d(\partial_{abc}\phi_{ij})-\partial_a(\partial_{bcd}\phi_{ij})$, and are found to have the form

$0=2\partial_{bcd}T_{ija}+2\partial_{ijd}T_{abc}-\partial_{jbd}T_{cai}-\partial_{jcd}T_{abi}-\partial_{ibd}T_{caj}-\partial_{icd}T_{abj}$ minus the same expression with a, d interchanged. (32)

This is symmetric to the interchange of i with j, to the interchange of b with c and to the pairwise interchange of ij with bc, but antisymmetric to the interchange of a with d.

Another set of partial differential equations for V is obtained by eliminating ϕ_i from equation (28). Because of the formal similarity between equations (18) and (28), the process of elimination gives similar equations in the two cases. In place of equation (22) we now have

$$V_{ik:jl}\equiv-\partial_{\alpha\alpha}(\partial_{kl}\phi_{ij}-\partial_{il}\phi_{jk}-\partial_{kj}\phi_{il}+\partial_{ij}\phi_{lk})+\partial_{kl}U_{ij}-\partial_{il}U_{jk}-\partial_{jk}U_{il}+\partial_{ij}U_{lk}=0. \tag{33}$$

Using (31), the first line of this equation can be written as

$$-2(\partial_{\alpha kl}T_{ij\alpha}-\partial_{\alpha il}T_{jk\alpha}-\partial_{\alpha jk}T_{il\alpha}+\partial_{\alpha ij}T_{kl\alpha}).$$

Omitting a common factor and writing $\phi_{ijk\alpha}\partial_\alpha V$ for T_{ijk} and $\phi_{ij\alpha}\partial_\alpha V+2\phi_{ij\alpha\beta}\partial_{\alpha\beta}V$ for

U_{ij}, we obtain the equations in the final form

$$\begin{aligned}&\partial_{kl}(\phi_{ij\alpha}\partial_\alpha V)-\partial_{il}(\phi_{jk\alpha}\partial_\alpha V)-\partial_{jk}(\phi_{il\alpha}\partial_\alpha V)+\partial_{ij}(\phi_{lk\alpha}\partial_\alpha V)\\&\quad-2[\partial_{kl}(\partial_\beta\phi_{ij\alpha\beta}\,.\,\partial_\alpha V)-\partial_{il}(\partial_\beta\phi_{jk\alpha\beta}\,.\,\partial_\alpha V)-\partial_{jk}(\partial_\beta\phi_{il\alpha\beta}\,.\,\partial_\alpha V)\\&\quad+\partial_{ij}(\partial_\beta\phi_{kl\alpha\beta}\,.\,\partial_\alpha V)]=0.\end{aligned}\tag{34}$$

The highest-order terms again cancel and the order of the equations is reduced to three. Like $S_{ik:jl}$ the tensor $V_{ik:jl}$ has all the symmetries of the Riemann tensor and, besides, satisfies the Bianchi identities and the cyclic relations.

Using (31), ϕ_{ij} can be expressed as a triple integral over V and the known functions ϕ_{ijkl}. In the same manner, ϕ_i can be expressed as a double integral over V, ϕ_{ijkl}, ϕ_{ijk}, ϕ_{ij}. For arbitrary n the process can be continued until the last function ϕ^0 is reached.

5. The anisotropic oscillator

To illustrate the use of the above theory let us apply it to one of the simplest systems, the anisotropic oscillator. In a search for the constants of motion of this system we shall restrict ourselves to the case $n=3$, $m=2$, for which detailed calculations have been carried out by Majumdar and Englefield (1977).

For an operator of the type

$$L=\phi_1p_1^3+\phi_2p_1^2p_2+\phi_3p_1p_2^2+\phi_4p_2^3+\phi_5p_1^2+\phi_6p_1p_2+\phi_7p_2^2+\phi_8p_1+\phi_9p_2+\phi\tag{35}$$

commuting with the Hamiltonian $H=-\frac{1}{2}(p_1^2+p_2^2)+V(x,y)$, the solution of the equations (5.1), (5.2) yields (Majumdar and Englefield 1977)

$$\begin{aligned}&\phi_{111}=\phi_1=ay^3+by^2+cy+d\\&3\phi_{112}=\phi_2=-3axy^2-2bxy-cx-fy^2-hy-k\\&3\phi_{122}=\phi_3=3ax^2y+bx^2+2fxy+hx+gy+l\\&\phi_{222}=\phi_4=-ax^3-fx^2-gx-e,\end{aligned}\tag{36}$$

$$\begin{aligned}&\phi_{11}=\phi_5=-3axy-bx+my^2+(q+f)y+r\\&2\phi_{12}=\phi_6=3ax^2-2mxy-3ay^2+(n-b)y-qx+s\\&\phi_{22}=\phi_7=3axy+mx^2-nx+fy-r,\end{aligned}\tag{37}$$

where the coefficients of the powers of x, y are all arbitrary. In the case of the anisotropic oscillator with $V=\frac{1}{2}(Ax^2+By^2)$, $A\neq B$, equations (26), (25) (equations (2.9), (2.10) of I) reduce to

$$8mxy(A-B)+x(2f+q)(4A-B)+y(b+n)(-A+4B)+(g-c-s)(A-B)=0$$

and

$$-15axy(A-B)+bx(-5A+2B)+fy(-2A+5B)-h(A-B)=0.$$

These equations can hold for $B\neq 4A$ only if $a=h=m=2f+q=g-c-s=0$. Furthermore, consistency of the equation (5.4) (equation (2.7) of I) requires that $b=f=0$. Thus

$$a=b=f=h=m=q=g-c-s=0.$$

Retaining the other coefficients, we have from equations (5.3) and (5.4) (equations (2.6), (2.7) of I) the following expressions for ϕ_8, ϕ_9, ϕ:

$$\phi_8 = -\tfrac{3}{2}x^2(cy+d)A + y(cx^2/2+kx)B - (\tfrac{7}{6}gy^3+ly^2)B + (gy^3/6+\tfrac{1}{2}ly^2)A + \alpha y + \beta, \quad (38)$$

$$\phi_9 = -x(gy^2/2+ly)A + \tfrac{3}{2}y^2(gx+e)B + (\tfrac{7}{6}cx^3+kx^2)A - (cx^3/6+kx^2/2)B - \alpha x + \gamma, \quad (39)$$

$$\phi = -\tfrac{1}{2}xy(3c+g)A + \tfrac{1}{2}y(3gx+cx+3e+k)B + ry^2B - rx^2A - \tfrac{1}{2}nxy^2(A-2B) - \tfrac{3}{2}xdA - \tfrac{1}{2}xlA. \quad (40)$$

These expressions substituted in (5.5) give

$$xA(-n/2+\beta) + yB\gamma + \alpha xy(A-B) + \tfrac{1}{2}kx^2y(4A-B)B + \tfrac{1}{2}lxy^2(A-4B)A + x^3dA^2 + \tfrac{3}{2}y^3eB^2 - \tfrac{1}{6}cx^3y(A-B)(9A-B) + \tfrac{1}{6}gxy^3(A-B)(A-9B) = 0. \quad (41)$$

Constants of motion can now be determined by taking only one of the coefficients r, n, l, c, g at a time to be different from zero. The following constants of motion are thus obtained:

Non-zero coefficient	Frequency ratio	Constant of motion
r	arbitrary	$L_c = p_1^2 - p_2^2 - Ax^2 + By^2$
n	2:1	$L_p = xp_2^2 - yp_1p_2 - \tfrac{1}{2}p_1 + Bxy^2$
l	2:1	$L = p_1p_2^2 + By^2p_1 - 4Bxyp_2 - 2Bx$
g	3:1	$L_3 = yp_1p_2^2 - xp_2^3 + p_1p_2 + \tfrac{1}{3}By^3p_1 - 3Bxy^2p_2 - 3Bxy.$ (42)

As their existence implies the separability of the Schrödinger equation (Winternitz *et al* 1966) in the respective coordinate systems, L_c may be called the Cartesian and L_p the parabolic constant of motion. L_c is the difference of two independent constants of motion, $H_1 = p_1^2 - Ax^2$, $H_2 = p_2^2 - By^2$. The commutators of the constants of motion for the frequency ratio 2:1 are

$$[L_c, L_p] = 4L, \qquad [L_c, L] = 16BL_p,$$
$$[L_p, L] = -H_2^2 + 2H_1H_2 - 3B \equiv L_k, \qquad [L_c, L_k] = 0,$$
$$[L, L_k] = 8B(8HL_p + 3H_1L_p + 3L_pH_1),$$
$$[L_p, L_k] = 2(8HL + 3H_1L + 3LH_1).$$

Since three of the six commutators are quadratic, not linear, functions of L_c, L_p, L, L_k, these operators do not generate a closed Lie algebra (Englefield 1972 § 1.1).

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Appendix 1

Equations (5) are obtained by separately equating to zero the coefficients of all distinct products of the p's in the expression for $[H, L]$. To find the nature of the second term of the equations let us consider the product $p_{i_1} \dots p_{i_{n-r}}$ associated with the $(r+2)$th equation of the set. If i_1 occurs α_1 times, i_2 occurs α_2 times, ... etc. in it, then, under the permutation of the i's, the second term of the equation develops into a sum, while the other terms, being symmetric, are multiplied by the factor $(n-r)!/(\alpha_1! \dots \alpha_{n-r}!)$. The terms in the sum with $i_1, i_2, \dots$ etc. as indices of differentiation are, respectively,

$$(\alpha_1, \dots, \alpha_{n-r})(n-r-1)!/(\alpha_1! \dots \alpha_{n-r}!)$$

in number. Permuting the indices in $\partial_{i_r}\phi_{i_1 \dots i_{r-1} i_{r+1} \dots i_{n-r}}$ is, therefore, equivalent to taking every index once to be an index of differentiation and multiplying the series so developed by $(n-r-1)!/(\alpha_1! \dots \alpha_{n-r}!)$. When this is done and a factor $(n-r)!/(\alpha_1! \dots \alpha_{n-r}!)$ is taken out the equations assume precisely the form (5).

Next, we prove the assertion made in § 3 that the coefficient tensor is a multiple of the derivative tensor. Solutions of equations (5.1) and (5.2) can be expressed tensorially in the form

$$\phi_{i_1 \dots i_p} = (i_1 \dots i_p \mid a_1 \dots a_q) x_{a_1} \dots x_{a_q}, \qquad \text{with } p = n, n-1; \, q \leq p. \tag{A1.1}$$

When terms of the same kind arising from the q-fold summation on the right-hand side are added together, the expression assumes the form

$$\phi_{i_1 \dots i_p} = \sum [q!/\beta_1! \dots \beta_q!](i_1 \dots i_p \mid a_1 \dots a_q) x_{a_1} \dots x_{a_q}, \tag{A1.2}$$

summed over distinct products of the x's. Here, a_1 occurs β_1 times, a_2 occurs β_2 times, and so on. Differentiating (A1.2) β_1 times with respect to x_{a_1}, β_2 times with respect to x_{a_2}, etc. gives

$$\partial_{a_1 \dots a_q} \phi_{i_1 \dots i_p} = q!(i_1 \dots i_p \mid a_1 \dots a_q). \tag{A1.3}$$

Appendix 2

In solution (36) the non-zero components of the coefficient tensor have the values

$(111|222) = a$, $(111|22) = b$, $(111|2) = c$, $(112|122) = -a/3$,

$(112|12) = -b/3$, $(112|22) = -f/3$, $(112|1) = -c/3$,

$(112|2) = -h/3$, $(122|112) = a/3$, $(122|11) = b/3$, $(122|12) = f/3$,

$(122|1) = h/3$, $(122|2) = g/3$, $(222|111) = -a$, $(222|11) = -f$,

$(222|1) = -g$. (A2.1)

It is easily verified that these coefficients satisfy the symmetry relations

$$(ijk \mid abc) = -(abc \mid ijk) \tag{A2.2a}$$

$$(ijk \mid abc) = (iab \mid jkc) + (jab \mid kic) + (kab \mid ijc) \tag{A2.2b}$$

$$(ijk \mid abc) = -(ija \mid kbc) - (jka \mid ibc) - (kia \mid jbc) \tag{A2.2c}$$

$$(ijk \mid ab) = (iab \mid jk) + (jab \mid ki) + (kab \mid ij) \tag{A2.2d}$$

$$(ijk\,|\,ab) = -(ija\,|\,kb) - (jka\,|\,ib) - (kia\,|\,jb) \tag{A2.2e}$$

$$(ijk\,|\,a) = -(ija\,|\,k) - (jka\,|\,i) - (kia\,|\,j). \tag{A2.2f}$$

References

Bander M and Itzykson C 1966 *Rev. Mod. Phys.* **38** 330, 346
Bargmann V 1936 *Z. Phys.* **99** 576
Barut A O, Budini P and Fronsdal C 1966 *Proc. R. Roc.* A **291** 106
Barut A O and Kleinert H 1967 *Phys. Rev.* **160** 1149
Bethe H A and Leon M 1962 *Phys. Rev.* **127** 636
Burgatti P 1911 *Atti Acad. Naz. Lincei Rc.* **20** 108
Dall'Aqua F A 1908 *Math. Ann.* **66** 398
—— 1912 *Rc. Circ. Mat. Palermo* **33** 341
Dietz W 1976 *J. Phys. A: Math. Gen.* **9** 519
Dothan Y, Gell-Mann M and Neeman Y 1965 *Phys. Rev. Lett.* **17** 148
Eisenhart L P 1934 *Ann. Math.* **35** 284
—— 1948 *Phys. Rev.* **74** 87
Englefield M J 1972 *Group Theory and the Coulomb Problem* (New York: Wiley)
Flamand G 1966 *J. Math. Phys.* **7** 1924
Fock V 1935 *Z. Phys.* **98** 145
Fris J, Mandrosov V, Smorodinsky Ya A, Uhlir M and Winternitz P 1965 *Phys. Lett.* **16** 354
Havas P 1975a *J. Math. Phys.* **16** 1461
—— 1975b *J. Math. Phys.* **16** 2476
Hughes J W B 1967 *Proc. Phys. Soc.* **91** 810
Joseph A 1967 *Rev. Mod. Phys.* **39** 829
Lenz W 1924 *Z. Phys.* **24** 197
Levi-Civita T 1904 *Math. Ann.* **59** 383
Majumdar S D and Basu D 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 787
Majumdar S D and Englefield M J 1977 *Int. J. Theor. Phys.* **16** 829
Makarov A A, Smorodinsky J A, Valiev Kh and Winternitz P 1967 *Nuovo Cim.* **52**A 1061
Park D 1960 *Z. Phys.* **159** 155
Pauli W 1926 *Z. Phys.* **36** 336
Stäckel P 1890 *Math. Ann.* **35** 91
—— 1891 *Über die Integration der Hamilton–Jacobi'schen Differentialgleichung mittels Separation der Variablen* (Halle: Habilitations Schrift)
—— 1893 *Math. Ann.* **42** 537
Swamy N V V J and Biedenharn L C 1964 *Phys. Rev.* **133** B 353
Winternitz P, Smorodinsky Ya A, Uhlir M and Fris I 1966 *Yadernaya Fizika* **4** 625 (Transl. 1967 *Soviet J. Nucl. Phys.* **4** 444)